ΟΜΟΙΟΜΟΡΦΗ ΣΥΝΕΧΕΙΑ

Ορισμός: Η $f: \Delta(f) \to \mathbb{R}$ θα λέγεται ομοιόμορφα συνεχής στο $\Delta(f)$

$\iff (\forall \varepsilon>0)(\exists \delta>0)(\forall x, y \in \Delta(f)) : |x-y|<\delta \Rightarrow |f(x)-f(y)| < \varepsilon$.

Παράδειγμα 4.41

ΝΔΟ η $f(x)=x^2$, $x \in [0,1]$ ομοιόμορφα συνεχής.

ΛΥΣΗ

Πρέπει $(\forall \varepsilon>0)(\exists \delta>0)(\forall x, y \in [0,1]) : |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$$|f(x)-f(y)| = |x^2-y^2| = |x-y||x+y| < \delta |x+y| < 2\delta$$

$$\left.\begin{array}{l} \bullet\ 0 \le x \le 1 \\ \bullet\ 0 \le y \le 1 \end{array}\right\} 0 \le x+y \le 2$$

όπου $\delta = \dfrac{\varepsilon}{2}$.

Παράδειγμα 4.42

ΝΔΟ η $f(x)=\eta\mu x$, $x \in \mathbb{R}$ ομοιόμορφα συνεχής.

ΛΥΣΗ

Πρέπει $(\forall \varepsilon>0)(\exists \delta>0)(\forall x, y \in \mathbb{R}) : |x-y|<\varepsilon \Rightarrow |f(x)-f(y)|<\varepsilon$

$$|f(x)-f(y)| = |\eta\mu x - \eta\mu y| = \left|2\,\eta\mu \frac{x-y}{2} \cdot \sigma\upsilon\nu \frac{x+y}{2}\right| \le \cancel{2} \cdot \frac{|x-y|}{\cancel{2}} = |x-y| = \varepsilon$$

ΑΣΚΗΣΕΙΣ

4.33) ΝΔΟ οι συναρτήσεις είναι ομοιόμορφα συνεχείς

α. $f(x)=x^3$, $-1 \le x \le 1$ δ. $f(x) = \dfrac{x}{x^2+1}$, $-2 \le x \le 0$

β. $f(x)=x^4$, $-3 \le x \le 2$ ε. $f(x) = \sqrt{x}$, $1 \le x \le 4$

γ. $f(x) = \dfrac{x}{x+2}$, $|x| \le 1$ στ. $f(x) = \dfrac{1}{\sqrt{x}}$, $x \ge 1$.

ΛΥΣΗ

α. $(\forall \varepsilon>0)(\exists \delta>0)(\forall x, y \in [-1,1]) : |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$$|f(x)-f(y)| = |x^3-y^3| = |x-y||x^2+xy+y^2| < \delta |x^2+yx+y^2| \quad (1)$$

$$\left.\begin{array}{l} -1 \le x \le 1 \\ -1 \le y \le 1 \end{array}\right\} \text{πολ/ζω} \quad 1 \le xy \le 1, \quad xy=1$$

και $-1 \le x \le 1 \Rightarrow 1 \le x^2 \le 1 \Rightarrow x^2=1$

$-1 \le y \le 1 \Rightarrow 1 \le y^2 \le 1 \Rightarrow y^2=1$

Άρα η (1) $< \delta \cdot 3$

Άρα $\boxed{\delta = \dfrac{\varepsilon}{3}}$

β) $(\forall \varepsilon>0)(\exists \delta>0)(\forall y,x\in[-3,2]) : |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon.$

$$|f(x)-f(y)| = |x^4-y^4| = |x^2-y^2||x^2+y^2| = |x-y||x+y||x^2+y^2| \quad ①.$$

$$\left.\begin{array}{l} \bullet\ -3\le x\le 2 \\ \bullet\ -3\le y\le 2 \end{array}\right\} -6\le x+y\le 4 \Rightarrow |x+y|\le 4$$

$$\left.\begin{array}{l} \bullet\ -3\le x\le 2 \Rightarrow 4\le x^2\le 9 \\ \bullet\ -3\le y\le 2 \Rightarrow 4\le y^2\le 9 \end{array}\right\} |x^2+y^2|\le 18$$

Άρα ① $< \delta\cdot 4\cdot 18 = \delta\cdot 72$, Συνεπώς $\delta = \dfrac{\varepsilon}{72}$

γ) $(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\in[-1,1]) : |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$$|f(x)-f(y)| = \left|\frac{x}{x+2}-\frac{y}{y+2}\right| = \left|\frac{x(y+2)-y(x+2)}{(x+2)(y+2)}\right| =$$

$$= \frac{|\cancel{xy}+2x-\cancel{xy}-2y|}{|(x+2)(y+2)|} = \frac{2|x-y|}{|x+2||y+2|} \quad ①$$

$$\left.\begin{array}{l} \bullet\ -1\le x\le 1 \Rightarrow 1\le x+2\le 3 \Rightarrow \frac{1}{3}\le \frac{1}{x+2}\le 1 \\ \bullet\ -1\le y\le 1 \Rightarrow 1\le y+2\le 3 \Rightarrow \frac{1}{3}\le \frac{1}{y+2}\le 1 \end{array}\right\} \quad \frac{1}{9}\le \frac{1}{(x+2)(y+2)}\le 1$$

Άρα ① $< \dfrac{2\delta}{1}$, Άρα $\delta = \dfrac{\varepsilon}{2}$.

δ) $(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\in[-2,0]) : |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon.$

$$|f(x)-f(y)| = \left|\frac{x}{x^2+1}-\frac{y}{y^2+1}\right| = \left|\frac{x(y^2+1)-y(x^2+1)}{(x^2+1)(y^2+1)}\right| =$$

$$= \frac{|xy^2+x-yx^2-y|}{|x^2+1||y^2+1|} = \frac{|(x-y)-xy(x-y)|}{|x^2+1||y^2+1|} = \frac{|x-y||1-xy|}{(1+x^2)(1+y^2)} \quad ①.$$

$$\left.\begin{array}{l} \bullet\ -2\le x\le 0 \Rightarrow 0\le x^2\le 4 \Rightarrow 1\le x^2+1\le 5 \Rightarrow \frac{1}{5}\le \frac{1}{x^2+1}\le 1 \\ \bullet\ -2\le y\le 0 \Rightarrow 0\le y^2\le 4 \Rightarrow 1\le y^2+1\le 5 \Rightarrow \frac{1}{5}\le \frac{1}{y^2+1}\le 1 \end{array}\right\} \frac{1}{x^2+1}\cdot\frac{1}{y^2+1}\le 1$$

$$\bullet\ 0\le xy\le 4 \Rightarrow -4\le -xy\le 0 \Rightarrow -3\le 1-xy\le 1$$

① $< \delta\cdot 1\cdot 1 < \varepsilon \Leftrightarrow \delta=\varepsilon$

ε) $(\forall \varepsilon>0)(\exists \delta>0)(\forall y,x\in[1,4]): |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$.

$|f(x)-f(y)| = |\sqrt{x}-\sqrt{y}| = \dfrac{|x-y|}{\sqrt{x}+\sqrt{y}}$ ① $\forall x,y\in[1,4]$.

- $1\le x\le 4 \Rightarrow 1\le \sqrt{x}\le 2$
- $1\le y\le 4 \Rightarrow 1\le \sqrt{y}\le 2$

$\Bigg\}\ 2\le \sqrt{x}+\sqrt{y}\le 4 \Rightarrow \dfrac{1}{4}\le \dfrac{1}{\sqrt{x}+\sqrt{y}}\le \dfrac{1}{2}$

Άρα, ① $< \delta\cdot\dfrac{1}{2} = \dfrac{\delta}{2} < \varepsilon$ άρα $\underline{\delta=2\varepsilon}$.

στ) $(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\in[1,+\infty)): |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$|f(x)-f(y)| = \left|\dfrac{1}{\sqrt{x}}-\dfrac{1}{\sqrt{y}}\right| = \dfrac{|\sqrt{y}-\sqrt{x}|}{\sqrt{x}\cdot\sqrt{y}} = \dfrac{|x-y|}{(\sqrt{x}\cdot\sqrt{y})(\sqrt{y}+\sqrt{x})}$ ①

$x\ge 1 \Rightarrow \sqrt{x}\ge 1$

$y\ge 1 \Rightarrow \sqrt{y}\ge 1$

$\Bigg\}\ \boxed{\sqrt{x}\cdot\sqrt{y}\ge 1} \Rightarrow \dfrac{1}{\sqrt{x}\cdot\sqrt{y}}\le 1$

$\Bigg\}\ \boxed{\sqrt{x}+\sqrt{y}\ge 2} \Rightarrow \dfrac{1}{\sqrt{x}+\sqrt{y}}\le \dfrac{1}{2}$

· Άρα, ① $< \delta\cdot 1\cdot\dfrac{1}{2} < \varepsilon \Rightarrow \underline{\delta=2\varepsilon}$.

4.34) ΝΔΟ οι συναρτήσεις είναι ομοιόμορφα συνεχείς:

i) $f(x)=\dfrac{1}{x},\ x\ge 1$

ii) $f(x)=\dfrac{1}{1+x^2},\ x\in\mathbb{R}$

iii) $f(x)=\dfrac{x}{1+x^2},\ x\in\mathbb{R}$

iv) $f(x)=\dfrac{x}{1+|x|},\ x\in\mathbb{R}$

<u>ΛΥΣΗ</u>

i) $(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\ge 1): |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

$|f(x)-f(y)| = \left|\dfrac{1}{x}-\dfrac{1}{y}\right| = \dfrac{|x-y|}{x\cdot y}$ ①

$\left.\begin{array}{l}\cdot\, x\ge 1\\ \cdot\, y\ge 1\end{array}\right\} \Rightarrow x\cdot y\ge 1 \Rightarrow \dfrac{1}{x\cdot y}\le 1$

· Άρα, ①. $\delta\cdot 1<\varepsilon \Rightarrow \boxed{\delta=\varepsilon}$.

iii) $(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\in\mathbb{R}): |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

- $|f(x)-f(y)| = \left|\frac{x}{x^2+1} - \frac{y}{y^2+1}\right| = \frac{|x+xy^2-yx^2-y|}{(x^2+1)(y^2+1)} = \frac{|x-y|\,|1-xy|}{(1+x^2)(1+y^2)} \leq$

$\leq |x-y| \cdot \frac{1+|x|\cdot|y|}{(1+x^2)(1+y^2)}$ ①.

Έστω, ότι $\frac{1+|x|\cdot|y|}{(1+x^2)(1+y^2)} \leq 1 \Rightarrow 1+|x|\cdot|y| \leq (1+x^2)(1+y^2) \Rightarrow$

$\Rightarrow (1+x^2)(1+y^2) - (1+|x||y|) \geq 0 \Rightarrow$

$\Rightarrow 1+x^2+y^2+x^2y^2-1-|x|\cdot|y| \geq 0 \Rightarrow$

$\Rightarrow (1+x^2)|y|^2-|x||y|+x^2 \geq 0$

$\Delta = |x|^2 - 4(1+x^2)\cdot x^2 = x^2-4x^2-4x^4 = -3x^2-4x^4 \leq 0$ ισχύει

Άρα ① $\leq \delta\cdot 1<\varepsilon \Rightarrow \delta=\varepsilon$.

ii) $(\forall \varepsilon>0)(\exists \delta>0)(\forall x,y\in\mathbb{R}): |x-y|<\delta \Rightarrow |f(x)-f(y)|<\varepsilon$

- $|f(x)-f(y)| = \left|\frac{1}{x^2+1} - \frac{1}{y^2+1}\right| = \frac{|y^2+1-x^2-1|}{(x^2+1)(y^2+1)} = \frac{|x^2-y^2|}{(x^2+1)(y^2+1)} =$

$= \frac{|x-y|\,|x+y|}{(x^2+1)(y^2+1)} < \delta\cdot|x+y|\cdot\frac{1}{(x^2+1)(y^2+1)}$ ①

όπου $\frac{|x+y|}{(x^2+1)(y^2+1)} \leq \frac{|x|+|y|}{(x^2+1)(y^2+1)} = \underbrace{\frac{|x|}{(x^2+1)(y^2+1)}}_{\leq 1} + \underbrace{\frac{|y|}{(x^2+1)(y^2+1)}}_{\leq 1}$

$\frac{|x|}{(x^2+1)(y^2+1)} \leq 1 \Rightarrow |x| \leq (x^2+1)(y^2+1)$ ... ισχύει

$\frac{|y|}{(x^2+1)(y^2+1)} \leq 1 \Rightarrow |y| \leq (x^2+1)(y^2+1)$ ... ισχύει

Άρα η ① $<\delta\cdot 2$ άρα $\boxed{\delta=\frac{\varepsilon}{2}}$